

On Generalized Heawood Inequalities for Manifolds: a van Kampen–Flores-type Nonembeddability Result^{*†}

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Abstract

The fact that the complete graph K_5 does not embed in the plane has been generalized in two independent directions. On the one hand, the solution of the classical *Heawood problem* for graphs on surfaces established that the complete graph K_n embeds in a closed surface M (other than the Klein bottle) if and only if $(n-3)(n-4) \leq 6b_1(M)$, where $b_1(M)$ is the first \mathbb{Z}_2 -Betti number of M . On the other hand, van Kampen and Flores proved that the k -skeleton of the n -dimensional simplex (the higher-dimensional analogue of K_{n+1}) embeds in \mathbb{R}^{2k} if and only if $n \leq 2k+1$.

Two decades ago, Kühnel conjectured that the k -skeleton of the n -simplex embeds in a compact, $(k-1)$ -connected $2k$ -manifold with k th \mathbb{Z}_2 -Betti number b_k only if the following *generalized Heawood inequality* holds: $\binom{n-k-1}{k+1} \leq \binom{2k+1}{k+1} b_k$. This is a common generalization of the case of graphs on surfaces as well as the van Kampen–Flores theorem.

In the spirit of Kühnel’s conjecture, we prove that if the k -skeleton of the n -simplex embeds in a $2k$ -manifold with k th \mathbb{Z}_2 -Betti number b_k , then $n \leq 2b_k \binom{2k+2}{k} + 2k + 4$. This bound is weaker than the generalized Heawood inequality, but does not require the assumption that M is $(k-1)$ -connected. Our results generalize to maps without q -covered points, in the spirit of Tverberg’s theorem, for q a prime power. Our proof uses a result of Volovikov about maps that satisfy a certain homological triviality condition.

1 Introduction

Given a closed surface M , a natural question is to determine the maximum integer n such that the complete graph K_n can be embedded (drawn without crossings) into M (e.g., $n = 4$ if $M = S^2$ is the 2-sphere, and $n = 7$ if M is a torus). This classical problem was raised in the late 19th century by Heawood [Hea90] and Heffter [Hef91] and completely settled in the 1950–60’s through a sequence of works by Gustin, Guy, Mayer, Ringel, Terry, Welch, and Youngs (see [Rin74, Ch. 1] for a discussion of the history of the problem and detailed references). Heawood already observed that if K_n embeds into M then

$$(n-3)(n-4) \leq 6b_1(M) = 12 - 6\chi(M), \quad (1)$$

where $\chi(M)$ is the Euler characteristic of M and $b_1(M) = 2 - \chi(M)$ is the first \mathbb{Z}_2 -Betti number of M , i.e., the dimension of the first homology group $H_1(M; \mathbb{Z}_2)$ (here and throughout the paper, we work with

^{*}The work by Z.P. was partially supported by the Israel Science Foundation grant ISF-768/12. The work by Z.P. and M.T. was partially supported by the project CE-ITI (GACR P202/12/G061) of the Czech Science Foundation and by the ERC Advanced Grant No. 267165. Part of the research work of M.T. was conducted at IST Austria, supported by an *IST Fellowship*. The research of P.P. was supported by the ERC Advanced grant no. 320924. The work by I.M. and U.W. was supported by the Swiss National Science Foundation (grants SNSF-200020-138230 and SNSF-PP00P2-138948).

[†]An extended abstract of this work was presented at the 31st International Symposium on Computational Geometry [GMP⁺15].

homology with \mathbb{Z}_2 -coefficients). Conversely, for surfaces M other than the Klein bottle, the inequality is tight, i.e., K_n embeds into M if and only if (1) holds; this is a hard result, the bulk of the monograph [Rin74] is devoted to its proof. (The exceptional case, the Klein bottle, has $b_1 = 2$, but does not admit an embedding of K_7 , only of K_6 .)¹

The question naturally generalizes to higher dimension: Let $\Delta_n^{(k)}$ denote the k -skeleton of the n -simplex, the natural higher-dimensional generalization of $K_{n+1} = \Delta_n^{(1)}$ (by definition $\Delta_n^{(k)}$ has $n+1$ vertices and every subset of at most $k+1$ vertices forms a face). Given a $2k$ -dimensional manifold M , what is the largest n such that $\Delta_n^{(k)}$ embeds (topologically) into M ? This line of enquiry started in the 1930's when van Kampen [vK32] and Flores [Flo33] showed that $\Delta_{2k+2}^{(k)}$ does not embed into \mathbb{R}^{2k} (the case $k=1$ corresponding to the non-planarity of K_5). Somewhat surprisingly, little else seems to be known, and the following conjecture of Kühnel [Küh94, Conjecture B] regarding a *generalized Heawood inequality* remains unresolved:

Conjecture 1 (Kühnel). *Let $n, k \geq 1$ be integers. If $\Delta_n^{(k)}$ embeds in a compact, $(k-1)$ -connected $2k$ -manifold M with k th \mathbb{Z}_2 -Betti number $b_k(M)$ then*

$$\binom{n-k-1}{k+1} \leq \binom{2k+1}{k+1} b_k(M). \quad (2)$$

The classical Heawood inequality (1) and the van Kampen–Flores Theorem correspond the special cases $k=1$ and $b_k=0$, respectively. Kühnel states Conjecture 1 in slightly different form, in terms of Euler characteristic of M rather than $b_k(M)$; our reformulation is equivalent. The \mathbb{Z}_2 -coefficients are not important in the statement of the conjecture but they are convenient for our new developments.

1.1 New results toward Kühnel's conjecture

In this paper, our main result is an estimate, in the spirit of the generalized Heawood inequality (2), on the largest n such that $|\Delta_n^{(k)}|$ almost embeds into a given $2k$ -dimensional manifold. An almost embedding is a relaxation of the notion of embedding that is useful in setting up our proof method.

Let K be a finite simplicial complex and let $|K|$ be its underlying space (geometric realization). We define an *almost-embedding* of K into a (Hausdorff) topological space X to be a continuous map $f: |K| \rightarrow X$ such that any two disjoint simplices $\sigma, \tau \in K$ have disjoint images, $f(|\sigma|) \cap f(|\tau|) = \emptyset$. In particular, every embedding is an almost-embedding as well. Let us stress, however, that the condition for being an almost-embedding depends on the actual simplicial complex (the triangulation), not just the underlying space. That is, if K and L are two different complexes with $|K| = |L|$ then a map $f: |K| = |L| \rightarrow X$ may be an almost-embedding of K into X but not an almost-embedding of L into X . Our main result is the following.

Theorem 2. *If $\Delta_n^{(k)}$ almost embeds into a $2k$ -manifold M then*

$$n \leq 2 \binom{2k+2}{k} b_k(M) + 2k + 4,$$

where $b_k(M)$ is the k th \mathbb{Z}_2 -Betti number of M .

This bound is weaker than the conjectured generalized Heawood inequality (2) and is clearly not optimal (as we already see in the special cases $k=1$ or $b_k(M)=0$).

Apart from applying more generally to almost embeddings, the hypotheses of Theorem 2 are weaker than those of Conjecture 1 in that we do not assume the manifold M to be $(k-1)$ -connected. We conjecture that this connectedness assumption is not necessary for Conjecture 1, i.e., that (2) holds

¹The inequality (1), which by a direct calculation is equivalent to $n \leq c(M) := \lfloor (7 + \sqrt{1 + 24\beta_1(M)})/2 \rfloor$, is closely related to the *Map Coloring Problem* for surfaces (which is the context in which Heawood originally considered the question). Indeed, it turns out that for surfaces M other than the Klein bottle, $c(M)$ is the maximum chromatic number of any graph embeddable into M . For $M = S^2$ the 2-sphere (i.e., $b_1(M) = 0$), this is the *Four-Color Theorem* [AH77, AHK77]; for other surfaces (i.e., $b_1(M) > 0$) this was originally stated (with an incomplete proof) by Heawood and is now known as the *Map Color Theorem* or *Ringel–Youngs Theorem* [Rin74]. Interestingly, for surfaces $M \neq S^2$, there is a fairly short proof, based on edge counting and Euler characteristic, that the chromatic number of any graph embeddable into M is at most $c(M)$ (see [Rin74, Thms. 4.2 and 4.8]) whereas the hard part is the tightness of (1).

whenever $\Delta_n^{(k)}$ almost embeds into a $2k$ -manifold M . The intuition is that $\Delta_n^{(k)}$ is $(k-1)$ -connected and therefore the image of an almost-embedding cannot “use” any parts of M on which nontrivial homotopy classes of dimension less than k are supported.

Previous work. The following special case of Conjecture 1 was proved by Kühnel [Küh94, Thm. 2] (and served as a motivation for the general conjecture): Suppose that P is an n -dimensional simplicial convex polytope, and that there is a subcomplex of the boundary ∂P of P that is k -Hamiltonian (i.e., that contains the k -skeleton of P) and that is a triangulation of M , a $2k$ -dimensional manifold. Then inequality (2) holds. To see that this is indeed a special case of Conjecture 1, note that ∂P is a *piecewise linear* (PL) sphere of dimension $n-1$, i.e., ∂P is combinatorially isomorphic to some subdivision of $\partial\Delta_n$ (and, in particular, $(n-2)$ -connected). Therefore, the k -skeleton of P , and hence M , contains a subdivision of $\Delta_n^{(k)}$ and is $(k-1)$ -connected.

In this special case and for $n \geq 2k+2$, equality in (2) is attained if and only if P is a simplex. More generally, equality is attained whenever M is a triangulated $2k$ -manifold on $n+1$ vertices that is $(k+1)$ -neighborly (i.e., any subset of at most $k+1$ vertices forms a face, in which case $\Delta_n^{(k)}$ is a subcomplex of M). Some examples of $(k+1)$ -neighborly $2k$ -manifolds are known, e.g., for $k=1$ (the so-called *regular cases* of equality for the Heawood inequality [Rin74]), for $k=2$ [KL83, KB83] (e.g., a 3-neighborly triangulation of the complex projective plane) and for $k=4$ [BK92], but in general, a characterization of the higher-dimensional cases of equality for (2) (or even of those values of the parameters for which equality is attained) seems rather hard (which is maybe not surprising, given how difficult the construction of examples of equality already is for $k=1$).

1.2 Generalization to points covered q times

Kühnel’s conjecture can be recast in a broader setting suggested by a generalization by Sarkaria [Sar91, Thm 1.5] of the van Kampen–Flores Theorem. Sarkaria’s theorem states that if q is a prime, and d and k integers such that $d \leq \frac{q}{q-1}k$, then for every continuous map $f: |\Delta_{qk+2q-2}^{(k)}| \rightarrow \mathbb{R}^d$ there are q pairwise disjoint simplices $\sigma_1, \dots, \sigma_q \in K$ with intersecting images $f(|\sigma_1|) \cap \dots \cap f(|\sigma_q|) \neq \emptyset$. Sarkaria’s result was generalized by Volovikov [Vol96] for q being a prime power.

Define an *almost q -embedding* of K into a (Hausdorff) topological space X as a continuous map $f: |K| \rightarrow X$ such that any q pairwise disjoint faces $\sigma_1, \dots, \sigma_q \in K$ have disjoint images $f(|\sigma_1|) \cap \dots \cap f(|\sigma_q|) = \emptyset$. (So almost 2-embeddings are almost embeddings.) Again, being an almost q -embedding depends on the actual simplicial complex (the triangulation), not just the underlying space. Our proof technique also gives an estimate for almost q -embeddings when q is a prime power.

Theorem 3. *Let $q = p^m$ be a prime power. If $\Delta_n^{(k)}$ q -almost-embeds into a d -manifold M with $d \leq \frac{q}{q-1}k$ then*

$$n \leq ((q-2)k + 2q - 2) \binom{qk + 2q - 2}{k} b_k(M) + (2q-2)k + 4q - 4,$$

where $b_k(M)$ is the k th \mathbb{Z}_p -Betti number of M .

Theorem 3 specializes for $q=2$ to Theorem 2.

1.3 Proof technique

Our proof of Theorem 3 strongly relies on a different generalization of the van Kampen–Flores Theorem, due to Volovikov [Vol96] regarding maps into general manifolds but under an additional homological triviality condition.

Theorem 4 (Volovikov). *Let $q = p^m$ be a prime power. Let $f: \Delta_{qk+2q-2}^{(k)} \rightarrow M$ be a continuous map where M is a compact d -manifold with $d \leq \frac{q}{q-1}k$. If the induced homomorphism*

$$f_*: H_k(\Delta_{qk+2q-2}^{(k)}; \mathbb{Z}_p) \rightarrow H_k(M; \mathbb{Z}_p)$$

is trivial then f is not a q -almost embedding.

Theorem 4 is obtained by specializing the corollary in Volovikov's article [Vol96] to $m = d$ and $s = k + 1$. Note that Volovikov [Vol96] formulates the triviality condition in terms of cohomology, i.e., he requires that $f^* : H^k(M; \mathbb{Z}_p) \rightarrow H^k(\Delta_{2k+2}^{(k)}; \mathbb{Z}_p)$ is trivial. However, since we are working with field coefficients and the (co)homology groups in question are finitely generated, the homological triviality condition (which is more convenient for us to work with) and the cohomological one are equivalent.² Note that the homological triviality condition is automatically satisfied if $H_k(M; \mathbb{Z}_p) = 0$, e.g., if $M = \mathbb{R}^{2k}$ or $M = S^{2k}$. On the other hand, without the homological triviality condition, the assertion is in general not true for other manifolds (e.g., K_5 embeds into every closed surface different from the sphere, or $\Delta_8^{(2)}$ embeds into the complex projective plane).

The key idea of our approach is to show that if n is large enough and f is a mapping from $\Delta_n^{(k)}$ to M , then there is a q -almost-embedding g from $\Delta_s^{(k)}$ to $|\Delta_n^{(k)}|$ for some prescribed value of s such that the composed map $f \circ g : \Delta_s \rightarrow M$ satisfies Volovikov's condition. More specifically, the following is our main technical lemma:

Lemma 5. *Let $k, s \geq 1$ and $b \geq 0$ be integers. Let p be a prime number. There exists a value $n_0 := n_0(k, b, s, p)$ with the following property. Let $n \geq n_0$ and let f be a mapping of $|\Delta_n^{(k)}|$ into a manifold M with k th \mathbb{Z}_p -Betti number at most b . Then there exists a subdivision D of $\Delta_s^{(k)}$ and a simplicial map $g_{\text{simp}} : D \rightarrow \Delta_n^{(k)}$ with the following properties.*

1. *The induced map on the geometric realizations $g : |D| = |\Delta_s^{(k)}| \rightarrow |\Delta_n^{(k)}|$ is an almost-embedding from $\Delta_s^{(k)}$ to $|\Delta_n^{(k)}|$.*
2. *The homomorphism $(f \circ g)_* : H_k(\Delta_s^{(k)}; \mathbb{Z}_p) \rightarrow H_k(M; \mathbb{Z}_p)$ is trivial (see Section 2 below for the precise interpretation of $(f \circ g)_*$).*

The value n_0 can be taken as $\binom{s}{k}b(s - 2k) + 2s - 2k + 1$.

Therefore, if $s \geq qk + 2q - 2$, then $f \circ g$ cannot be a q -almost embedding by Volovikov's theorem. We deduce that f is not a q -almost-embedding either, and Theorem 3 immediately follows. This deduction requires the following lemma (proven in Section 2) as in general, a composition of a q -almost-embedding and an almost-embedding is not always a q -almost-embedding.

Lemma 6. *Let K and L be simplicial complexes and X a topological space. Suppose g is an almost embedding of K into $|L|$ and f is a q -almost embedding of L into X for some integer $q \geq 2$. Then $f \circ g$ is a q -almost embedding of K into X , provided that g is the realization of a simplicial map g_{simp} from some subdivision K' of K to L .*

Remark 7. The third author proved in his thesis [Pat15] a slightly better bound on n_0 in Lemma 5, namely $n_0 = \binom{s}{k}b(s - 2k) + s + 1$. The proof, however, uses colorful version of Lemma 12. Since the proof of the colorful version is long and technical and in the end it only improves the bound in Theorem 2 by 2, we have decided to present the more accessible version of the argument.

Paper organization. Before we establish Lemma 5 (in Section 4), thus completing the proof of Theorem 3, we first prove a weaker version that introduces the main ideas in a simpler setting, and yields a weaker bound for n_0 , stated in Equation (4). The reader interested only in the case $q = 2$ may want to consult a preliminary version of this paper [GMP⁺15] tailored to that case (where homology computations are without signs and the construction of the subdivision D is simpler).

2 Preliminaries

We begin by fixing some terminology and notation. We will use $\text{card}(U)$ to denote the cardinality of a set U .

²More specifically, by the Universal Coefficient Theorem [Mun84, 53.5], $H_k(\cdot; \mathbb{Z}_p)$ and $H^k(\cdot; \mathbb{Z}_p)$ are dual vector spaces, and f^* is the adjoint of f_* , hence triviality of f_* implies that of f^* . Moreover, if the homology group $H_k(X; \mathbb{Z}_p)$ of a space X is finitely generated (as is the case for both $\Delta_n^{(k)}$ and M , by assumption) then it is (non-canonically) isomorphic to its dual vector space $H^k(X; \mathbb{Z}_p)$. Therefore, f_* is trivial if and only if f^* is.

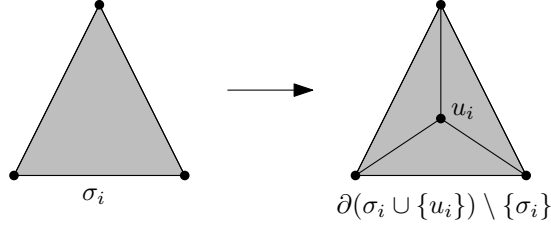


Figure 1: A stellar subdivision of a simplex.

We recall that the *stellar subdivision* of a maximal face ϑ in a simplicial complex K is obtained by removing ϑ from K and adding a cone $a_\vartheta * (\partial\vartheta)$, where a_ϑ is a newly added vertex, the apex of the cone (see Figure 1).

Throughout this paper we only work with homology groups and Betti numbers over \mathbb{Z}_p , and for simplicity, we mostly drop the coefficient group \mathbb{Z}_p from the notation. Moreover, we will need to switch back and forth between singular and simplicial homology. More precisely, if K is a simplicial complex then $H_*(K)$ will mean the simplicial homology of K , whereas $H_*(X)$ will mean the singular homology of a topological space X . In particular, $H_*(|K|)$ denotes the singular homology of the underlying space $|K|$ of a complex K . We use analogous conventions for $C_*(K)$, $C_*(X)$ and $C_*(|K|)$ on the level of chains, and likewise for the subgroups of cycles and boundaries, respectively.³ Given a cycle c , we denote by $[c]$ the homology class it represents.

A mapping $h: |K| \rightarrow X$ induces a chain map $h_\#^{\text{sing}}: C_*(|K|) \rightarrow C_*(X)$ on the level of singular chains; see [Hat02, Chapter 2.1]. There is also a canonical chain map $\iota_K: C_*(K) \rightarrow C_*(|K|)$ inducing the isomorphism of $H_*(K)$ and $H_*(|K|)$, see again [Hat02, Chapter 2.1]. We define $h_\#: C_*(K) \rightarrow C_*(X)$ as $h_\# := h_\#^{\text{sing}} \circ \iota_K$. The three chain maps mentioned above also induce maps h_*^{sing} , $(\iota_K)_*$, and h_* on the level of homology satisfying $h_* = h_*^{\text{sing}} \circ (\iota_K)_*$. We need a technical lemma saying that our maps compose, in a right way, on the level of homology.

Lemma 8. *Let K and L be simplicial complexes and X a topological space. Let j_{simp} be a simplicial map from K to L , $j: |K| \rightarrow |L|$ the continuous map induced by j_{simp} and $h: |L| \rightarrow X$ be another continuous map. Then $h_* \circ (j_{\text{simp}})_* = (h \circ j)_*$ where $(j_{\text{simp}})_*: H_*(K) \rightarrow H_*(L)$ is the map induced by j_{simp} on the level of simplicial homology and the maps h_* and $(h \circ j)_*$ are as defined above.*

Proof. The proof follows from the commutativity of the diagram below.

$$\begin{array}{ccccc}
 & & (h \circ j)_* & & \\
 & & \curvearrowright & & \\
 & & (h \circ j)_*^{\text{sing}} & & \\
 & & \curvearrowright & & \\
 H_*(|K|) & \xrightarrow{j_*^{\text{sing}}} & H_*(|L|) & \xrightarrow{h_*^{\text{sing}}} & H_*(X) \\
 \uparrow (\iota_K)_* & & \uparrow (\iota_L)_* & & \nearrow h_* \\
 H_*(K) & \xrightarrow{(j_{\text{simp}})_*} & H_*(L) & &
 \end{array}$$

The commutativity of the lower right triangle follows from the definition of h_* . Similarly $(h \circ j)_* = (h \circ j)_*^{\text{sing}} \circ (\iota_K)_*$. The fact that $(h \circ j)_*^{\text{sing}} = h_*^{\text{sing}} \circ j_*^{\text{sing}}$ follows from functoriality of the singular homology. The commutativity of the square follows from the naturality of the equivalence of the singular and simplicial homology; see [Mun84, Thm 34.4]. \square

We now prove the final technical step of our approach, stated in the introduction.

³We remark that throughout this paper, we will only work with spaces that are either (underlying spaces of) simplicial complexes or topological manifolds. Such spaces are homotopy equivalent to CW complexes [Mil59, Corollary 1], and so on the matter of homology, it does not really matter which (ordinary, i.e., satisfying the dimension axiom) homology theory we use as they are all naturally equivalent for CW complexes [Hat02, Thm. 4.59]. However the distinction between the simplicial and the singular setting will be relevant on the level of chains.

Proof of Lemma 6. Let $\sigma_1, \dots, \sigma_q$ be q pairwise disjoint faces of K . Our task is to show $f \circ g(|\sigma_1|) \cap \dots \cap f \circ g(|\sigma_q|) = \emptyset$. Let ϑ_i be a face of K' that subdivides σ_i for $i \in [q]$. We are done, if we prove

$$f \circ g(|\vartheta_1|) \cap \dots \cap f \circ g(|\vartheta_q|) = \emptyset \quad (3)$$

for every such possible choice of $\vartheta_1, \dots, \vartheta_q$.

The faces $\vartheta_1, \dots, \vartheta_q$ are pairwise disjoint since $\sigma_1, \dots, \sigma_q$ are pairwise disjoint. Since g_{simp} is a simplicial map inducing an almost embedding, the faces $g_{\text{simp}}(\vartheta_1), \dots, g_{\text{simp}}(\vartheta_q)$ are pairwise disjoint faces of L . Consequently, (3) follows from the fact that f is a q -almost embedding. \square

3 Proof of Lemma 5 with a weaker bound on n_0

Let k, b, s be fixed integers. We consider a $2k$ -manifold M with k th Betti number b , a map $f : |\Delta_n^{(k)}| \rightarrow M$. Recall that although we want to build an almost-embedding, homology is computed over \mathbb{Z}_p . The strategy of our proof of Lemma 5 is to start by designing an auxiliary chain map

$$\varphi : C_* \left(\Delta_s^{(k)} \right) \rightarrow C_* \left(\Delta_n^{(k)} \right).$$

that behaves as an almost-embedding, in the sense that whenever σ and σ' are disjoint k -faces of Δ_s , $\varphi(\sigma)$ and $\varphi(\sigma')$ have disjoint supports, and such that for every $(k+1)$ -face τ of Δ_s the homology class $[(f_{\#} \circ \varphi)(\partial\tau)]$ is trivial. We then use φ to design a subdivision D of $\Delta_s^{(k)}$ and a simplicial map $g_{\text{simp}} : D \rightarrow \Delta_n^{(k)}$ that induces a map $g : |D| \rightarrow |\Delta_n^{(k)}|$ with the desired properties: g is an almost-embedding and $(f \circ g)_*([\partial\tau])$ is trivial for all $(k+1)$ -faces τ of Δ_s . Since the cycles $\partial\tau$, for $(k+1)$ -faces τ of Δ_s , generate all k -cycles of $\Delta_s^{(k)}$, this implies that $(f \circ g)_*$ is trivial.

The purpose of this section is to give a first implementation of the above strategy that proves Lemma 5 with a bound of

$$n_0 \geq \left(\binom{s+1}{k+1} - 1 \right) p^{b \binom{s+1}{k+1}} + s + 1. \quad (4)$$

In Section 4 we then improve this bound to $\binom{s}{k} b(s-2k) + 2s - 2k + 1$ at the cost of some technical complications (note that the improved bound is independent of p).

Throughout the rest of this paper we use the following notations. We let $\{v_1, v_2, \dots, v_{n+1}\}$ denote the set of vertices of Δ_n and we assume that Δ_s is the induced subcomplex of Δ_n on $\{v_1, v_2, \dots, v_{s+1}\}$. We let $U = \{v_{s+2}, v_{s+3}, \dots, v_{n+1}\}$ denote the set of vertices of Δ_n *unused* by Δ_s . We let $m = \binom{s+1}{k+1}$ and denote by $\sigma_1, \sigma_2, \dots, \sigma_m$ the k -faces of Δ_s , ordered lexicographically.

Later on, when working with homology, we compute the simplicial homology with respect to this fixed order on the vertices of Δ_n . In particular, the boundary of a j -simplex $\vartheta = \{v_{i_1}, v_{i_2}, \dots, v_{i_{j+1}}\}$ where $i_1 \leq i_2 \leq \dots \leq i_{j+1}$ is

$$\partial\vartheta = \sum_{\ell=1}^{j+1} (-1)^{\ell+1} \vartheta \setminus \{v_{i_\ell}\}.$$

3.1 Construction of φ

For every face ϑ of Δ_s of dimension at most $k-1$ we set $\varphi(\vartheta) = \vartheta$. We then “route” each σ_i by mapping it to its stellar subdivision with an apex $u \in U$, *i.e.* by setting $\varphi(\sigma_i)$ to $\sigma_i + (-1)^k z(\sigma_i, u)$ where $z(\sigma_i, u)$ denotes the cycle $\partial(\sigma_i \cup \{u\})$; see Figure 2 for the case $k=1$.

We ensure that φ behaves as an almost-embedding by using a different apex $u \in U$ for each σ_i . The difficulty is to choose these m apices in a way that $[f_{\#}(\varphi(\partial\tau))]$ is trivial for every $(k+1)$ -face τ of Δ_s . To that end we associate to each $u \in U$ the sequence

$$\mathbf{v}(u) := ([f_{\#}(z(\sigma_1, u))], [f_{\#}(z(\sigma_2, u))], \dots, [f_{\#}(z(\sigma_m, u))]) \in H_k(M)^m,$$

and we denote by $\mathbf{v}_i(u)$ the i th element of $\mathbf{v}(u)$. We work with \mathbb{Z}_p -homology, so $H_k(M)^m$ is finite; more precisely, its cardinality equals p^{bm} . From $n \geq n_0 = (m-1)p^{bm} + s + 1$ we get that $\text{card}(U) \geq (m-1) \text{card}(H_k(M)^m) + 1$.

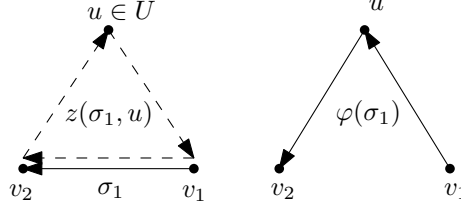


Figure 2: Rerouting σ_i for $k = 1$. The support of $z(\sigma_i, u)$ is dashed on the left, and the support of resulting $\varphi(\sigma_i)$ is on the right.

The pigeonhole principle then guarantees that there exist m distinct vertices u_1, u_2, \dots, u_m of U such that $\mathbf{v}(u_1) = \mathbf{v}(u_2) = \dots = \mathbf{v}(u_m)$. We use u_i to “route” σ_i and put

$$\varphi(\sigma_i) := \sigma_i + (-1)^k z(\sigma_i, u_i). \quad (5)$$

We finally extend φ linearly to $C_*\left(\Delta_s^{(k)}\right)$.

Lemma 9. *The map φ is a chain map and $[f_\#(\varphi(\partial\tau))] = 0$ for every $(k+1)$ -face $\tau \in \Delta_s$.*

Before proving the lemma, we establish a simple claim that will also be useful later.

Claim 10. *Let τ be a $(k+1)$ -face of Δ_s and let $u \in U$. Let $\sigma_{i_1}, \dots, \sigma_{i_{k+2}}$ be all the k -faces of τ sorted lexicographically, that is, $i_1 \leq \dots \leq i_{k+2}$. Then*

$$\partial\tau = z(\sigma_{i_1}, u) - z(\sigma_{i_2}, u) + \dots + (-1)^{k+1} z(\sigma_{i_{k+2}}, u). \quad (6)$$

Proof. This follows from expanding the equation $0 = \partial^2(\tau \cup \{u\})$. Indeed,

$$\begin{aligned} 0 &= \partial^2(\tau \cup \{u\}) = \partial(\sigma_{i_{k+2}} \cup \{u\} - \sigma_{i_{k+1}} \cup \{u\} + \dots + (-1)^{k+1} \sigma_{i_1} \cup \{u\} + (-1)^{k+2} \tau) \\ &= (-1)^{k+1} (-\partial\tau + z(\sigma_{i_1}, u) - z(\sigma_{i_2}, u) + \dots + (-1)^{k+1} z(\sigma_{i_{k+2}}, u)). \end{aligned}$$

□

Proof of Lemma 9. The map φ is the identity on ℓ -chains with $\ell \leq k-1$ and Equation (5) immediately implies that $\partial\varphi(\sigma) = \partial\sigma$ for every k -simplex σ . It follows that φ is a chain map.

Now let τ be a $(k+1)$ -simplex of Δ_s and let $\sigma_{i_1}, \dots, \sigma_{i_{k+2}}$ be its k -faces. We have

$$\begin{aligned} f_\# \circ \varphi(\partial\tau) &= f_\# \circ \varphi \left(\sum_{j=1}^{k+2} (-1)^{k+j} \sigma_{i_j} \right) = f_\# \left(\sum_{j=1}^{k+2} (-1)^{j+k} (\sigma_{i_j} + (-1)^k z(\sigma_{i_j}, u_{i_j})) \right) \\ &= f_\#(\partial\tau) + \sum_{j=1}^{k+2} (-1)^j f_\#(z(\sigma_{i_j}, u_{i_j})). \end{aligned}$$

$[f_\#(z(\sigma_{i_j}, u_\ell))] = \mathbf{v}_{i_j}(u_\ell)$ is independent of the value ℓ . When passing to the homology classes in the above identity, we can therefore replace each u_{i_j} with u_1 , and obtain,

$$[f_\# \circ \varphi(\partial\tau)] = [f_\#(\partial\tau)] + \sum_{j=1}^{k+2} (-1)^j [f_\#(z(\sigma_{i_j}, u_1))] = \left[f_\# \left(\partial\tau + \sum_{j=1}^{k+2} (-1)^j z(\sigma_{i_j}, u_1) \right) \right].$$

This class is trivial by Claim 10. Figure 3 illustrates the geometric intuition behind this proof. □

t

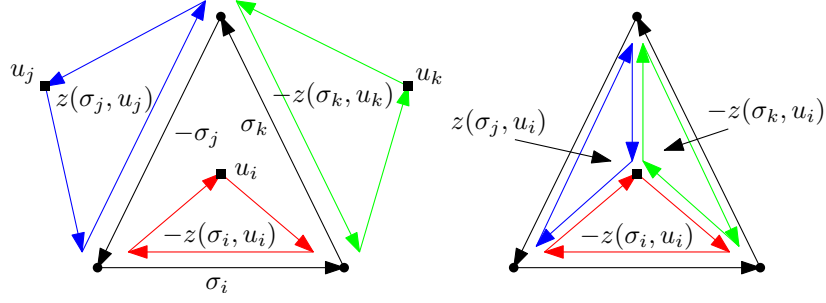


Figure 3: The geometric intuition behind the proof of Lemma 9, for $k = 1$ and $u_{i_1} = u_1$ (cycles of same color are in the same homology class; the class on the right is trivial, because the edges cancel out in pairs).

3.2 Subdivisions and orientations

Our next task is the construction of D and g ; however, we first mention a few properties of subdivisions.

Let us consider a simplicial complex K and a subdivision S of K . (So K and S are regarded as geometric simplicial complexes, and for every simplex η of S there is a simplex ϑ of K such that $\eta \subseteq \vartheta$. In this case, we say that η *subdivides* ϑ .) There is a canonical chain map $\rho: C_*(K) \rightarrow C_*(S)$ that induces an isomorphism in homology. Intuitively, ρ maps a simplex ϑ of K to a sum of simplices of S of the same dimension that subdivide ϑ . However, we have to be careful about the ± 1 coefficients in the sum.

We work with the ordered simplicial homology, that is, we order the vertices of K as well as the vertices of S . We want to define the mutual orientation $\text{or}(\eta, \vartheta) \in \{-1, 1\}$ of a j -simplex ϑ of K and a j -simplex η of S that subdivides ϑ . We set up $\text{or}(\eta, \vartheta)$ to be 1 if the orientations of ϑ and η agree, and -1 if they disagree; the orientation of each geometric simplex is computed relative to the order of its vertices in K or S (with respect to a fixed base of their common affine hull, say). Then we set

$$\rho(\vartheta) = \sum_{\eta} \text{or}(\eta, \vartheta) \eta \quad (7)$$

where the sum is over all simplices η in S of the same dimension as ϑ which subdivide ϑ . Finally, we extend ρ to a chain map. It is routine to check that ρ commutes with the boundary operator and that it induces an isomorphism on homology. It is also useful to describe ρ in the specific case where S is a stellar subdivision of a complex K consisting of a single k -simplex. Here, we assume that w_1, \dots, w_{j+1} are the vertices of K in this order (in K as well as in S) and a is the apex of S , which comes last in the order on S . We also consider S as a subcomplex of the $(k+1)$ -simplex on w_1, \dots, w_{k+1}, a . And we use the notation $z(\vartheta, a) = \partial(\vartheta \cup \{a\})$, analogously as previously in the case of k -faces of Δ_s .

Lemma 11. *In the setting above, let ϑ be the k -face of K . Then $\rho(\vartheta) = \vartheta + (-1)^k z(\vartheta, a)$.*

Proof. Let $\eta_i := \vartheta \cup \{a\} \setminus \{w_i\}$ for $i \in [k+1]$. Then η_i are all faces of S subdividing ϑ . We have $\text{or}(\eta_i, \vartheta) = (-1)^{i+k+1}$ as η_i has the same orientation as ϑ with respect to a modified order of vertices of ϑ obtained by replacing w_i with a . Therefore $\rho(\vartheta) = \sum_{i=1}^{k+1} (-1)^{i+k+1} \eta_i$. On the other hand,

$$z(\vartheta, a) = \partial(\vartheta \cup \{a\}) = \left(\sum_{i=1}^{k+1} (-1)^{i+1} \eta_i \right) + (-1)^{k+3} \vartheta = (-1)^k (\rho(\vartheta) - \vartheta).$$

□

3.3 Construction of D and g

The definition of φ , and in particular Equation (5), suggests to construct our subdivision D of $\Delta_s^{(k)}$ by simply replacing every k -face of $\Delta_s^{(k)}$ by its stellar subdivision. Let a_i denote the new vertex introduced

when subdividing σ_i . We fix a linear order on vertices of D in such a way that we reuse the order of vertices that also belong to $\Delta_s^{(k)}$ and then the vertices a_i follow in arbitrary order.

We define a simplicial map $g_{\text{simp}}: D \rightarrow \Delta_n^{(k)}$ by putting $g_{\text{simp}}(v) = v$ for every original vertex v of $\Delta_s^{(k)}$, and $g_{\text{simp}}(a_i) = u_i$ for $i \in [m]$. This g_{simp} induces a map $g: |\Delta_s^{(k)}| \rightarrow |\Delta_n^{(k)}|$ on the geometric realizations. Since the u_i 's are pairwise distinct, g is an embedding⁴, so Condition 1 of Lemma 5 holds.

In principle, we would like to derive Condition 2 of Lemma 5 by observing that g ‘induces’ a chain map from $C_*(\Delta_s^{(k)})$ to $C_*(\Delta_n^{(k)})$ that coincides with φ . Making this a formal statement is thorny because g , as a continuous map, naturally induces a chain map $g_\#$ on singular rather than simplicial chains. We can't use directly g_{simp} either, since we are interested in a map from $C_*(\Delta_s^{(k)})$ and not from $C_*(D)$.

We handle this technicality as follows. We consider the chain map $\rho: C_*(\Delta_s^{(k)}) \rightarrow C_*(D)$ from (7). This map induces an isomorphism ρ_* in homology. In addition $\varphi = (g_{\text{simp}})_\# \circ \rho$ where $(g_{\text{simp}})_\#: C_*(D) \rightarrow C_*(\Delta_n^{(k)})$ denotes the (simplicial) chain map induced by g_{simp} . Indeed, all three maps are the identity on simplices of dimension at most $k-1$. For a k -simplex σ , the map g_{simp} is an order preserving isomorphism when restricted to the subdivision of σ (in D). Therefore, the required equality $\varphi(\sigma) = (g_{\text{simp}})_\# \circ \rho(\sigma)$ follows from (5) and Lemma 11.

We thus have in homology

$$f_* \circ \varphi_* = f_* \circ (g_{\text{simp}})_* \circ \rho_*$$

and since ρ_* is an isomorphism and $f_* \circ \varphi_*$ is trivial by Lemma 9, it follows that $f_* \circ (g_{\text{simp}})_*$ is also trivial. Since $f_* \circ (g_{\text{simp}})_* = (f \circ g)_*$ by Lemma 8, $(f \circ g)_*$ is trivial as well. This concludes the proof of Lemma 5 with the weaker bound.

4 Proof of Lemma 5

We now prove Lemma 5 with the bound claimed in the statement, namely

$$n_0 = \binom{s}{k} b(s-2k) + 2s - 2k + 1.$$

Let k, b, s be fixed integers. We consider a $2k$ -manifold M with k th \mathbb{Z}_p -Betti number b , a map $f: |\Delta_n^{(k)}| \rightarrow M$, and we assume that $n \geq n_0$.

The proof follows the same strategy as in Section 3: we construct a chain map $\varphi: C_*(\Delta_s^{(k)}) \rightarrow C_*(\Delta_n^{(k)})$ such that the homology class $[(f_\# \circ \varphi)(\partial\tau)]$ is trivial for all $(k+1)$ -faces τ of Δ_s , then upgrade φ to a continuous map $g: |\Delta_s^{(k)}| \rightarrow |\Delta_n^{(k)}|$ with the desired properties.

When constructing φ , we refine the arguments of Section 3 to “route” each k -face using not only one, but several vertices from U ; this makes finding “collisions” easier, as we can use linear algebra arguments instead of the pigeonhole principle. This comes at the cost that when upgrading g , we must content ourselves with proving that it is an almost-embedding. This is sufficient for our purpose and has an additional benefit: the same group of vertices from U may serve to route several k -faces provided they pairwise intersect in $\Delta_k^{(s)}$.

4.1 Construction of φ

We use the same notation regarding v_1, \dots, v_{n+1} , Δ_n , Δ_s , U , $m = \binom{s+1}{k+1}$ and $\sigma_1, \sigma_2, \dots, \sigma_m$ as in Section 3.

Multipoints and the map \mathbf{v} . As we said we plan to route k -faces of Δ_s through certain collections of vertices from U (weighted); we will call these collections multipoints. It is more convenient to work with them on the level of formal linear combinations. Let $C_0(U)$ denote the \mathbb{Z}_p -vector space of formal linear combinations of vertices from U . A *multipoint* is an element of $C_0(U)$ whose coefficients sum to 1 (in \mathbb{Z}_p , of course). The multipoints form an affine subspace of $C_0(U)$ which we denote by \mathcal{M} . The *support*, $\text{sup}(\mu)$, of a multipoint $\mu \in \mathcal{M}$ is the set of vertices $v \in U$ with non-zero coefficient in μ . We say that two multipoints are *disjoint* if their supports are disjoint.

⁴We use the full strength of almost-embeddings when proving Lemma 5 with the better bound on n_0 .

For any k -face σ_i and any multipoint $\mu = \sum_{u \in U} \lambda_u u$ we define:

$$z(\sigma_i, \mu) := \sum_{u \in \text{sup}(\mu)} \lambda_u z(\sigma_i, u) := \sum_{u \in \text{sup}(\mu)} \lambda_u \partial(\sigma_i \cup \{u\}).$$

Now, we proceed as in Section 3 but replace unused points by multipoints of \mathcal{M} and the cycles $z(\sigma_i, u)$ with the cycles $z(\sigma_i, \mu)$. Since \mathbb{Z}_p is a field, $H_k(M)^m$ is a vector space and we can replace the sequences $\mathbf{v}(u)$ of Section 3 by the linear map

$$\mathbf{v} : \begin{cases} \mathcal{M} & \rightarrow H_k(M)^m \\ \mu & \mapsto ([f_{\#}(z(\sigma_1, \mu))], [f_{\#}(z(\sigma_2, \mu))], \dots, [f_{\#}(z(\sigma_m, \mu))]) \end{cases}$$

Finding collisions. The following lemma takes advantage of the vector space structure of $H_k(M)^m$ and the affine structure of \mathcal{M} to find disjoint multipoints μ_1, μ_2, \dots to route the σ_i 's more effectively than by simple pigeonhole.

Lemma 12. *For any $r \geq 1$, any \mathbb{Z}_p -vector space V , and any affine map $\psi: \mathcal{M} \rightarrow V$, if $\text{card}(U) \geq (\dim(\psi(\mathcal{M})) + 1)(r - 1) + 1$ then \mathcal{M} contains r disjoint multipoints $\mu_1, \mu_2, \dots, \mu_r$ such that $\psi(\mu_1) = \psi(\mu_2) = \dots = \psi(\mu_r)$.*

Proof. Let us write $U = \{v_{s+2}, v_{s+3}, \dots, v_{n+1}\}$ and $d = \dim(\psi(\mathcal{M}))$. We first prove by induction on r the following statement:

If $\text{card}(U) \geq (d + 1)(r - 1) + 1$ there exist r pairwise disjoint subsets $I_1, I_2, \dots, I_r \subseteq U$ whose image under ψ have affine hulls with non-empty intersection.

(This is, in a sense, a simple affine version of Tverberg's theorem.) The statement is obvious for $r = 1$, so assume that $r \geq 2$ and that the statement holds for $r - 1$. Let A denote the affine hull of $\{\psi(v_{s+2}), \psi(v_{s+3}), \dots, \psi(v_{n+1})\}$ and let I_r denote a minimal cardinality subset of U such that the affine hull of $\{\psi(v) : v \in I_r\}$ equals A . Since $\dim A \leq d$ the set I_r has cardinality at most $d + 1$. The cardinality of $U \setminus I_r$ is at least $(d + 1)(r - 2) + 1$ so we can apply the induction hypothesis for $r - 1$ to $U \setminus I_r$. We thus obtain $r - 1$ disjoint subsets I_1, I_2, \dots, I_{r-1} whose images under ψ have affine hulls with non-empty intersection. Since the affine hull of $\psi(U \setminus I_r)$ is contained in the affine hull of $\psi(I_r)$, the claim follows.

Now, let $a \in V$ be a point common to the affine hulls of $\psi(I_1), \psi(I_2), \dots, \psi(I_r)$. Writing a as an affine combination in each of these spaces, we get

$$a = \sum_{u \in J_1} \lambda_u^{(1)} \psi(u) = \sum_{u \in J_2} \lambda_u^{(2)} \psi(u) = \dots = \sum_{u \in J_r} \lambda_u^{(r)} \psi(u)$$

where $J_j \subseteq I_j$ and $\sum_{u \in J_j} \lambda_u^{(j)} = 1$ for any $j \in [r]$. Setting $\mu_j = \sum_{u \in J_j} \lambda_u^{(j)} u$ finishes the proof. \square

Computing the dimension of $\mathbf{v}(\mathcal{M})$. Having in mind to apply Lemma 12 with $V = H_k(M)^m$ and $\psi = \mathbf{v}$, we now need to bound from above the dimension of $\mathbf{v}(\mathcal{M})$. An obvious upper bound is $\dim H_k(M)^m$, which equals $bm = b \binom{s+1}{k+1}$. A better bound can be obtained by an argument analogous to the proof of Lemma 9. We first extend Claim 10 to multipoints.

Claim 13. *Let τ be a $(k + 1)$ -face of Δ_s and let $\mu \in \mathcal{M}$. Let $\sigma_{i_1}, \dots, \sigma_{i_{k+2}}$ be all the k -faces of τ sorted lexicographically. Then*

$$\partial \tau = z(\sigma_{i_1}, \mu) - z(\sigma_{i_2}, \mu) + \dots + (-1)^{k+1} z(\sigma_{i_{k+2}}, \mu). \quad (8)$$

Proof. By Claim 10 we know that (8) is true for points. For a multipoint $\mu = \sum_{u \in U} \lambda_u u$, we get (8) as a linear combination of equations (6) for the points u with the 'weight' λ_u (note that $\sum_{u \in U} \lambda_u = 1$; therefore the corresponding combination of the left-hand sides of (6) equals $\partial \tau$). \square

Lemma 14. $\dim(\mathbf{v}(\mathcal{M})) \leq b \binom{s}{k}$.

Proof. Let τ be a $(k+1)$ -face of Δ_s and let $\sigma_{i_1}, \dots, \sigma_{i_{k+2}}$ denote its k -faces. For any multipoint μ , Claim 13 implies

$$[f_{\#}(\partial\tau)] = \sum_{j=1}^{k+2} (-1)^{j+1} [f_{\#}(z(\sigma_{i_j}, \mu))] = \sum_{j=1}^{k+2} (-1)^{j+1} \mathbf{v}_{i_j}(\mu);$$

therefore

$$\mathbf{v}_{i_{k+2}}(\mu) = (-1)^{k+1} [f_{\#}(\partial\tau)] + \sum_{j=1}^{k+1} (-1)^{j+k+1} \mathbf{v}_{i_j}(\mu).$$

Each vector $\mathbf{v}(\mu)$ is thus determined by the values of the $\mathbf{v}_j(\mu)$'s where σ_j contains the vertex v_1 . Indeed, the vectors $[f_{\#}(\partial\tau)]$ are independent of μ , and for any σ_i not containing v_1 we can eliminate $\mathbf{v}_i(\mu)$ by considering $\tau := \sigma_i \cup \{v_1\}$ (and setting $\sigma_{i_{k+2}} = \sigma_i$). For each of the $\binom{s}{k}$ faces σ_j that contain v_1 , the vector $\mathbf{v}_j(\mu)$ takes values in $H_k(M)$ which has dimension at most b . It follows that $\dim \mathbf{v}(\mathcal{M}) \leq b \binom{s}{k}$. \square

Coloring graphs to reduce the number of multipoints used. We could now apply Lemma 12 with $r = m$ to obtain one multipoint per k -face, all pairwise disjoint, to proceed with our “routing”. As mentioned above, however, we only need that φ is an almost-embedding, so we can use the same multipoint for several k -faces provided they pairwise intersect. Optimizing the number of multipoints used reformulates as the following graph coloring problem:

Assign to each k -face σ_i of Δ_s some color $c(i) \in \mathbb{N}$ such that $\text{card}\{c(i) : 1 \leq i \leq m\}$ is minimal and disjoint faces use distinct colors.

This question is classically known as Kneser’s graph coloring problem and an optimal solution uses $s - 2k + 1$ colors [Lov78, Mat03]. Let us spell out one such coloring (proving its optimality is considerably more difficult, but we do not need to know that it is optimal). For every k -face σ_i we let $\min \sigma_i$ denote the smallest index of a vertex in σ_i . When $\min \sigma_i \leq s - 2k$ we set $c(i) = \min \sigma_i$, otherwise we set $c(i) = s - 2k + 1$. Observe that any k -face with color $c \leq s - 2k$ contains vertex v_c . Moreover, the k -faces with color $s - 2k + 1$ consist of $k + 1$ vertices each, all from a set of $2k + 1$ vertices. It follows that any two k -faces with the same color have some vertex in common.

Defining φ . We are finally ready to define the chain map $\varphi: C_*(\Delta_s^{(k)}) \rightarrow C_*(\Delta_n^{(k)})$. Recall that we assume that $n \geq n_0 = \left(\binom{s}{k}b + 1\right)(r - 1) + s + 1$. Using the bound of Lemma 14 we can apply Lemma 12 with $r = s - 2k + 1$, obtaining $s - 2k + 1$ multipoints $\mu_1, \mu_2, \dots, \mu_{s-2k+1} \in \mathcal{M}$. We set $\varphi(\vartheta) = \vartheta$ for any face ϑ of Δ_s of dimension less than k . We then “route” each k -face σ_i through the multipoint $\mu_{c(i)}$ by putting

$$\varphi(\sigma_i) := \sigma_i + (-1)^k z(\sigma_i, \mu_{c(i)}), \tag{9}$$

where $c(i)$ is the color of σ_i in the coloring of the Kneser graph proposed above. We finally extend φ linearly to $C_*(\Delta_s)$.

We need the following analogue of Lemma 9.

Lemma 15. *The map φ is a chain map and $[f_{\#}(\varphi(\partial\tau))] = 0$ for every $(k+1)$ -face $\tau \in \Delta_s$.*

The proof of Lemma 15 is very similar to the proof of Lemma 9; it just replaces points with multipoints and Claim 10 with Claim 13. We therefore omit the proof. We next argue that φ behaves like an almost embedding.

Lemma 16. *For any two disjoint faces ϑ, η of $\Delta_s^{(k)}$, the supports of $\varphi(\vartheta)$ and $\varphi(\eta)$ use disjoint sets of vertices.*

Proof. Since φ is the identity on chains of dimension at most $(k-1)$, the statement follows if neither face has dimension k . For any k -chain σ_i , the support of $\varphi(\sigma_i)$ uses only vertices from σ_i and from the support of $\mu_{c(i)}$. Since each $\mu_{c(i)}$ has support in U , which contains no vertex of Δ_s , the statement also holds when exactly one of ϑ or η has dimension k . When both ϑ and η are k -faces, their disjointness implies that they use distinct μ_j 's, and the statement follows from the fact that distinct μ_j 's have disjoint supports. \square

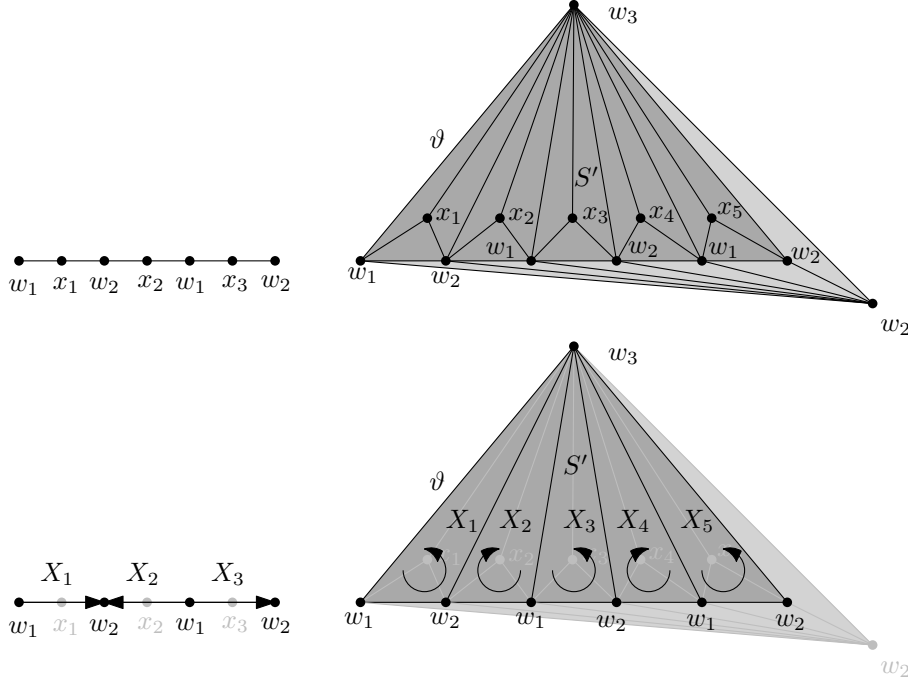


Figure 4: Examples of subdivisions for $k = 1$ and $\ell = 3$ (left) and for $k = 2$ and $\ell = 5$ (right). The bottom pictures show the orientations of $|X_i|$ in the given ordering.

4.2 Construction of D and g

We define D and g similarly as in Section 3, but the switch from points to multipoints requires to replace stellar subdivisions by a slightly more complicated decomposition.

The subdivision D . We define D so that it coincides with Δ_s on the faces of dimension at most $(k - 1)$ and decomposes each face of dimension k independently. The precise subdivision of a k -face σ_i depends on the cardinality of the support of the multipoint $\mu_{c(i)}$ used to “route” σ_i under φ , but the method is generic and spelled out in the next lemma; refer to Figure 4.

Lemma 17. *Let $k \geq 1$ and $\sigma = \{w_1, w_2, \dots, w_{k+1}\}$ be a k -simplex. For any positive odd integer $\ell \geq 1$ there exist a subdivision S of σ in which no face of dimension $k - 1$ or less is subdivided, and a labelling of the vertices of S by $\{w_1, w_2, \dots, w_{k+1}, x_1, x_2, \dots, x_\ell\}$ (some labels may appear several times) satisfying the following properties.*

1. *Every vertex in S corresponding to an original vertex w_i of σ is labelled by w_i .*
2. *No k -face of S has its vertices labelled by w_1, w_2, \dots, w_{k+1} ,*
3. *for every $j \in [\ell]$, the subdivision S contains exactly one vertex labelled by x_j ; this vertex appears in a copy X_j of a stellar subdivision of a simplex labelled by w_1, \dots, w_{k+1} with the apex labelled x_j .*
4. *Let us equip vertices of S with a linear order which respects the order $w_1 \leq w_2 \leq \dots \leq w_{k+1} \leq x_1 \leq \dots \leq x_\ell$ of the labels. For each $j \in [\ell]$ considering $|X_j|$ as a simplex in $|S| = |\sigma|$, such $|X_j|$ is oriented coherently with $|\sigma|$ (in the given ordering) if and only if j is odd.*

Proof. This proof is done in the language of geometric simplicial complexes (rather than abstract ones).

The case $\ell = 1$ can be done by a stellar subdivision and labelling the added apex x_1 . The case $k = 1$ is easy, as illustrated in Figure 4 (left). We therefore assume that $k \geq 2$ and build our subdivision and labelling in four steps:

- We start with the boundary of our simplex σ where each vertex w_i is labelled by itself. Let ϑ be the $(k-1)$ -face of $\partial\sigma$ opposite vertex w_2 , ie labelled by $w_1, w_3, w_4, \dots, w_{k+1}$. We create a vertex in the interior of σ , label it w_2 , and construct a new simplex σ' as the join of ϑ and this new vertex; this is the dark simplex in Figure 4 (right).
- We then subdivide σ' by considering $\ell-1$ distinct hyperplanes passing through the vertices of σ' labelled w_3, w_4, \dots, w_{k+1} and through an interior points of the edge of σ' labelled w_1, w_2 . These hyperplanes subdivide σ' into ℓ smaller simplices. We label the new interior vertices on the edge of σ' labelled w_1, w_2 by alternatively, w_1 and w_2 ; since ℓ is odd we can do so in a way that every sub-edge is bounded by two vertices labelled w_1, w_2 .
- We operate a stellar subdivision of each of the ℓ smaller simplices subdividing σ' , and label the added apices x_1, x_2, \dots, x_ℓ . This way we obtain a subdivision S' of σ' .
- We finally consider each face η of S' subdividing $\partial\sigma'$ and other than ϑ and add the simplex formed by η and the (original) vertex w_2 of σ . These simplices, together with S' , form the desired subdivision S of σ .

It follows from the construction that no face of $\partial\sigma$ was subdivided.

Property 1 is enforced in the first step and preserved throughout. We can ensure that Property 2 holds in the following way. First, we have that any k -simplex of S' contains a vertex x_j for some $j \in [\ell]$. Next, if we consider a k -simplex of S which is not in S' it is a join of a certain $(k-1)$ -simplex η of S' , with $\eta \subset \partial\sigma'$, and the vertex w_2 of σ . However, the only such $(k-1)$ -simplex labelled by $w_1, w_3, w_4, \dots, w_{k+1}$ is ϑ , but the join of ϑ and w_2 does not belong to S .

Properties 3 and 4 are enforced by the stellar subdivisions of the third step and by alternating the labels w_1 and w_2 in the second step. No other step creates, destroys or modifies any simplex involving a vertex labelled x_j . \square

Let S be the subdivision of a simplex σ from Lemma 17. Similarly as in the case of Lemma 11, we need to describe the chain map $\rho: C_*(\sigma) \rightarrow C_*(S)$ defined by formula (7). Actually, only a partial information will be sufficient for us, focusing on k -simplices of X_j .

Since for every $j \in [\ell]$, the apex of X_j is the only vertex labelled by x_j , we can use x_j as the name for the apex. Let ϑ_j be the k -simplex on the vertices of X_j except of x_j . Note that this simplex does not belong to S . Following the usual pattern, we also denote $z(\vartheta_j, x_j) := \partial(\vartheta_j \cup \{x_j\})$.

Lemma 18. *In the setting above,*

$$\rho(\sigma) = \sum_{j=1}^{\ell} (-1)^{j+1} (\vartheta_j + (-1)^k z(\vartheta_j, x_j)) + \sum_{\eta} \text{or}(\eta, \sigma) \eta \quad (10)$$

where the second sum is over all k -simplices of S which do not belong to any X_j .

Proof. We expand $\rho(\sigma)$ via (7); however, we further shift the k -simplices in some of the X_j to the first sum in (10). This is done via Lemma 11 on each of the X_j ; the correction term $(-1)^{j+1}$ comes from Property 4 of Lemma 17. \square

The subdivision D of $\Delta_s^{(k)}$ is now defined as follows. First, we leave the $(k-1)$ -skeleton untouched. Next for each k -simplex σ_i we consider the multipoint $\mu = \mu_{c(i)} = \sum_{u \in U} \lambda_u u$ (leaving the dependence on the index i implicit in the affine combination). We recall that λ_u are elements of \mathbb{Z}_p ; however, we temporarily consider them as elements of \mathbb{Z} , in the interval $\{0, 1, \dots, p-1\}$. We consider some $u' \in U$, which belongs to the support of μ , and we set $\kappa_u := \lambda_u$ for any $u \in U \setminus \{u'\}$ (as elements of \mathbb{Z}) whereas we set $\kappa_{u'} := 1 - \sum_{u \in U \setminus \{u'\}} \lambda_u$. It follows that $\kappa_u \equiv \lambda_u \pmod{p}$ for any $u \in U$ as $\sum_{u \in U} \lambda_u \equiv 1 \pmod{p}$ (they sum to 1 as elements of \mathbb{Z}_p). Next, we set $\ell_i := \sum_{u \in U} |\kappa_u|$. It follows that ℓ_i is odd, and we set $S(i)$ to be the subdivision of σ_i obtained from Lemma 17 with $\ell := \ell_i$. The final subdivision D is obtained by subdividing each σ_i this way. For working with the chains, we need to specify a global linear order on the vertices D . We pick an arbitrary such order that respects the prescribed order on each $S(i)$.

According to this subdivision, we have a chain map $\rho: C_*(\Delta_s^{(k)}) \rightarrow C_*(D)$ defined in Subsection 3.2. On faces of dimension at most $(k-1)$ it is an identity; on k -faces, it is determined by the formula from Lemma 18.

The simplicial map g_{simp} . We now define a simplicial map $g_{\text{simp}}: D \rightarrow \Delta_n^{(k)}$. We first set $g_{\text{simp}}(v) = v$ for every vertex v of Δ_s . Next, we consider some k -face $\sigma_i = \{w_1, \dots, w_{k+1}\}$. We denote by v_1, v_2, \dots, v_{k+1} the vertices on the boundary of $S(i)$, being understood that each v_j is labelled by w_j . We map each interior vertex of $S(i)$ labelled with w_j to v_j . It remains to map interior vertices of $S(i)$ labelled x_j for $j \in [\ell]$. Using the notation from the definition of D , we consider the integers κ_u for $u \in U$ (with respect to our σ_i). If $\kappa_u > 0$, then we pick κ_u vertices x_j with j odd and we map them to u . If $\kappa_u < 0$, which may happen only for $u = u'$ (coming again from the definition of D), then we pick $-\kappa_u$ vertices x_j with j even and we map them to u . Of course, for two distinct elements u_1 and u_2 from U we pick distinct points x_j . The parameter $\ell = \ell_i$ is set up exactly in such a way that we cover all x_j . Now we need to know that ρ and g_{simp} compose to φ on the level of chains.

Lemma 19. $(g_{\text{simp}})_\# \circ \rho = \varphi$.

Proof. All three maps are the identity on $\Delta_s^{(k-1)}$ so let us focus on the k -faces. Consider a k -face σ_i , the value $\rho(\sigma_i)$ is given by the formula in Lemma 18 with $S = S(i)$. However, for expressing $(g_{\text{simp}})_\# \circ \rho(\sigma_i)$ we may ignore the second sum in formula (10) since a k -simplex η of S that does not belong to any X_j contains two vertices with a same label by Lemma 17, which implies that $(g_{\text{simp}})_\#(\eta) = 0$.

Therefore

$$(g_{\text{simp}})_\# \circ \rho(\sigma_i) = (g_{\text{simp}})_\# \left(\sum_{j=1}^{\ell} (-1)^{j+1} (\vartheta_j + (-1)^k z(\vartheta_j, x_j)) \right) = \sum_{u \in U} \kappa_u (\sigma_i + (-1)^k z(\sigma_i, u)). \quad (11)$$

The last equality follows from the definition of g_{simp} considering that g_{simp} preserves the prescribed linear orders on D and $\Delta_n^{(k)}$. The sign $(-1)^{j+1}$ disappears as the vertices x_j with j even contribute to κ_u with the opposite sign. We know that $\kappa_u \bmod p = \lambda_u$ and that $\sum_{u \in U} \kappa_u = 1$. Therefore the expression on the right-hand side of (11) equals $\sigma_i + (-1)^k z(\sigma_i, \mu)$, that is, $\varphi(\sigma_i)$ as required. \square

The continuous map g . Since D is a subdivision of $\Delta_s^{(k)}$, we have $|\Delta_s^{(k)}| = |D|$ and the simplicial map $g_{\text{simp}}: D \rightarrow \Delta_n^{(k)}$ induces a continuous map $g: |\Delta_s^{(k)}| \rightarrow |\Delta_n^{(k)}|$. All that remains to do is check that g satisfies the two conditions of Lemma 5. Condition 1 follows from a direct translation of Lemma 16; note that in the definition of g_{simp} we map x_j to $u \in U$ only if $\kappa_u \neq 0$. Condition 2 can be verified by a computation in the same way as in Section 3. Specifically, in homology we have

$$f_* \circ \varphi_* = f_* \circ (g_{\text{simp}})_* \circ \rho_*$$

and we know that $f_* \circ \varphi_*$ is trivial on $\Delta_s^{(k)}$ by Lemma 15. As ρ_* is an isomorphism, this implies that $f_* \circ (g_{\text{simp}})_*$ is trivial. Lemma 8 then implies that $(f \circ g)_*$ is trivial. This concludes the proof of Lemma 5.

Acknowledgement. U.W. learned about Conjecture 1 from Wolfgang Kühnel when attending the *Mini Symposia on Discrete Geometry and Discrete Topology* at the *Jahrestagung der Deutschen Mathematiker-Vereinigung* in München in 2010. He would like to thank the organizers Frank Lutz and Achill Schürmann for the invitation, and Prof. Kühnel for stimulating discussions.

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